An experimentalist's attempt to understand why g=2 for an electron – or where does the electron's spin come from?

"A great deal more was hidden in the Dirac equation than the author had expected when he wrote it down in 1928. Dirac himself remarked in one of his talks that his equation was more intelligent than its author. It should be added, however, that it was Dirac who found most of the additional insights." Weisskopf on Dirac

The Dirac equation

At the start of the quantum world, it was not obvious how quantum mechanics con be reconciled with special relativity.

Initial ideas were based upon quantising the relativistic energy equation. (E,p) being an invariant four-vector

$$E^2 - p^2c^2 = m^2c^4 \tag{1.1}$$

With the following standard substitutions

$$E \rightarrow -i\hbar \frac{\partial}{\partial t}$$

and

$$\bar{p} \rightarrow -i\hbar \frac{\partial}{\partial x_i}$$

giving what is now known as the Klein-Gordon Equation

$$-\hbar \frac{\partial^2}{\partial t^2} - c^2 \hbar \frac{\partial^2}{\partial x_i^2} = m^2c^4 \tag{1.2}$$

Or $(\partial^2 + m^2)\Psi = 0$

The problem is that the Hamiltonian formed by (1.2) is

$$\sqrt{m^2c^4 - c^2\hbar\frac{\partial^2}{\partial x_i^2}}\Psi$$

which is not linear in special relativity co-ordinates.

Dirac (1928) postulated an equation which is a sort of square root of (1.1) – as much from a sense of beauty than anything else – i.e. that

$$\left[p_o - [\alpha.p + \beta.m]\right]\Psi = 0 \qquad (1.3)$$

Remembering that $E\Psi = H\Psi$, the Hamiltonian is $\alpha.p + \beta.m$ where α and β are 4 by 4 Hermitian matrices. (Note that we have moved to putting $\hbar = c = 1$). The equation is linear in both momentum and energy, so position and time are on the same footing, the requirement of special relativity.

Dirac's equation[1] can be expressed in a compact form as

$$(\mathrm{i}\gamma^u \partial_u - m)\Psi = 0 \qquad (1.4)$$

where $\partial_u = \frac{\partial}{\partial x_u}$ and u covers the range 0,1,2, 3 and γ^μ are the Dirac matrices, give by

$\gamma^0 = \beta, \gamma^i = \beta\alpha_i$ and

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

[Multiply (1.3) through by β, remembering that $\beta^2 = I$ or 1].

As a check on the form of the Dirac equation, we act on (1.4) with $(\mathrm{i}\gamma^u\partial_u + m)$ to give $-(\gamma^u\gamma^v\partial_u\partial_v + m^2)\Psi = 0$. Dirac's noticed that, as derivatives commute, this is the same as $(\frac{1}{2}\{\gamma^u, \gamma^v\}\partial_u\partial_v + m^2)\Psi$ and that $\{\gamma^u, \gamma^v\} = 2\eta^{uv}$ (η^{uv} being the "Minkowski tensor" – which sets out the metric of flat spacetime) his equation can be converted back to the Klein-Gordon equation, viz $(\partial^2 + m^2)\Psi = 0$. This requires that $(\gamma^0) = 1, (\gamma^j) = -1$ and that $\gamma^u\gamma^v = -\gamma^v\gamma^u$. This anti-commutation requirement remove the possibility the gamma could be ordinary numbers.

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[1] Feynman compacted Dirac's equation further by adding a bar to the partial derivative sign, $\not\partial$, to denote multiplication by $i\gamma^\mu$. Thus the equation became just $(\not\partial_u - m)\Psi = 0$

The Dirac equation for an electron in a magnetic field

We start with the basic Dirac equation

$$(\mathrm{i}\gamma^u\partial_u - m)\Psi = 0$$

and add an electromagnetic field requires us to move to a covariant derivative thus

$$\partial_u \to D_u = \partial_u - ieA_u$$

giving $(\mathrm{i}\gamma^u D_u - m)\Psi = 0$

Acting on this with $(i\gamma^u D_u + m)$ we have, given that the cross terms cancel,

$(\gamma^u\gamma^v D_u D_v + m^2)\Psi = 0$ but noting that the gamma matrices' commutators sum as follows

$\gamma^u\gamma^v = \frac{1}{2}\{\gamma^u,\gamma^v\} + \frac{1}{2}[\gamma^u,\gamma^v]$ and that

$\{\gamma^u,\gamma^v\} = 2\eta^{uv}$ as well as that $[\gamma^u,\gamma^v] = 2i\sigma^{uv}$ where $\sigma^{ij} = \varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}$

(η^{uv} is the Minkowski metric with signature (+,-,-,-))

Thus $\frac{1}{2}\{\gamma^u,\gamma^v\}D_u D_v = \frac{1}{2}(2\eta^{uv}D_u D_v) = D_u D^u$

So we are left with, for the Dirac equation in an electromagnetic field,

$$(D_u D^u + \frac{-\mathrm{i}\sigma^{uv}}{2}[D_u, D_v] + m^2)\Psi = 0$$

First turning our attention to the $[D_u, D_v]$ term: Expanding, we have

$$= (\partial_u - \mathrm{i}eA_u)(\partial_v - \mathrm{i}eA_v) - (\partial_v - \mathrm{i}eA_v)(\partial_u - \mathrm{i}eA_u)$$

$$= (\partial_u\partial_v - e^2A_uA_v - \mathrm{i}eA_u\partial_v - \mathrm{i}e\partial_uA_v) - (\partial_v\partial_u - e^2A_vA_u - \mathrm{i}eA_v\partial_u - \mathrm{i}e\partial_vA_u)$$

But we have missed something: The operators are actually acting on the wavefunction Ψ, so the ∂_uA_v and ∂_vA_u terms need to be expanded to $(\partial_uA_v) + A_v\partial_u$ and $(\partial_vA_u) + A_u\partial_v$ respectively, where the $()$ indicates that ∂_p only acts on A_q. This gives

$$(\partial_u\partial_v - e^2A_uA_v - ieA_u\partial_v - \mathrm{i}e((\partial_uA_v) + A_v\partial_u)) - (\partial_v\partial_u - e^2A_vA_u - \mathrm{i}eA_v\partial_u - \mathrm{i}e((\partial_vA_u) + A_u\partial_v))$$

Noting that derivatives commute, we are left with

$= (-ie(\partial_u A_v)) - (-ie(\partial_v A_u))$ or $= -ie(\partial_u A_v - \partial_v A_u) = -ieF_{uv}$ where F_{uv} is the electromagnetic tensor. So we now have

$(D_u D^u + \frac{-\mathrm{i}\sigma^{uv}}{2}(-\mathrm{i}e)F_{uv} + m^2)\Psi = 0$ Or

$$(D_u D^u - \frac{e\sigma^{uv}}{2}F_{uv} + m^2)\Psi = 0 \qquad (1.5)$$

The orbital angular momentum term

Now looking at the first term $D_u D^u$ and putting the system in a weak B_0 field[2] in the third (say z) axis [see note[1] below on this approximation. Then $A_0 = 0, A_1 = -\frac{1}{2}B_0 x^2, A_2 = \frac{1}{2}B_0 x^1, A_3 = 0$

Noting that

$$D_u D^u \equiv (D_i)^2 = (\partial_i)^2 - ie(\partial_i A_i - A_i \partial_i) + 0(A_i)^2$$

As the magnetic field is weak, we can ignore the $(A_i)^2$ term and again we must think of all of these operators as acting on Ψ. Thus $\partial_i A_i \equiv \partial_i A_i \Psi = A_i \partial_i \Psi + \Psi \partial_i A_i \equiv A_i \partial_i + (\partial_i A_i)$ where again the () indicates that ∂_i only acts on A_i. But in this case $\partial_i A_i = 0$ (by substitution into our weak field above) so $(\partial_i A_i - A_i \partial_i) = 2A_i \partial_i$. So $(D_i)^2 = (\partial_i)^2 - 2\mathrm{i}eA_i\partial_i + 0(A_i)^2$ or

$$(D_i)^2 = (\partial_i)^2 - 2\mathrm{i}eB_o\frac{1}{2}(x^1\partial_2 - x^2\partial_1) + 0(A_i)^2$$

giving us $(D_i)^2 = (\partial_i)^2 - 2ieB_o.\vec{x}\times\vec{p} + 0(A_i)^2$. We recall, of course, that $\vec{x}\times\vec{p} = \vec{L}$ so $(D_i)^2 = (\partial_i)^2 - ieB_o.\vec{L} + 0(A_i)^2$.

Here we see the classical orbital angular momentum term interacting with the B-field.

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[2] Checking the F_{12} component of the electromagnetic tensor, we have

$$F_{12} = \partial_1 A_2 - \partial_2 A_1$$

$$F_{12} = \partial_1(\frac{1}{2}B_0 x^1) - \partial_2(-\frac{1}{2}B_0 x^2),$$

we have $F_{12} = \frac{1}{2}B_0 - (-\frac{1}{2}B_0) = B_0$ as required.

The Classical Hamiltonian term

Recalling that the four component wavefunction can be dissolved into two components, so that $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ and noting that $\phi >> \chi$ in the non-relativistic limit (i.e. a slow electron): Why? Consider moving into a frame where the electron is stationary. Then we have $(m\gamma^0 - m)\Psi = 0$ after setting $p^u = (m, \vec{0})$. But

$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ so

$$m\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right)\Psi = 0$$

And compacting we have

$\begin{pmatrix} 0 & 0 \\ 0 & -2I \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0$ so $\chi = 0$ and ϕ can exist.

Writing $\phi = e^{-imt}\Psi$ we look at the $(\partial_0^2 - m^2)$ term in our equation.

Letting ϕ act on it, we have

$$\frac{\partial}{\partial_t}(e^{-imt}\Psi) = (-im + \frac{\partial}{\partial_t})e^{-imt}\Psi$$

And applying this again

$$\frac{\partial}{\partial_t}(-im + \frac{\partial}{\partial_t})e^{-imt}\Psi = (-im + \frac{\partial}{\partial_t})(-im + \frac{\partial}{\partial_t})e^{-imt}\Psi + \left(\frac{\partial}{\partial_t}\right)^2 e^{-imt}\Psi$$

Dropping terms in $\left(\frac{\partial}{\partial_t}\right)^2$ as being small and adding back the spatial derivatives we have for our $(\partial_0^2 - m^2)$ term, $(-m^2 - 2im\frac{\partial}{\partial_t} - \nabla^2 + m^2)e^{-imt}\Psi$

The electron Spin term

Returning to the full equation, we look at the middle term

$$\frac{-e}{2}\sigma^{uv}F_{uv}$$

and remembering that the system is in a weak B_0 field in the third axis and that $A_0 = 0, A_1 = -\frac{1}{2} B_0 x^2, A_2 = -\frac{1}{2} B_0 x^1, A_3 = 0$ we compute $\frac{e}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} F_{ij}$

Trying component 123 and 213 we have (watching the factors of two) $\frac{e}{2} \sigma^3 (F_{12} - F_{21})$

But $F_{12} = -F_{21} = B_o$ and noting that $\frac{\vec{\sigma}}{2} = \vec{S}$, the middle term becomes $2e\vec{B}.\vec{S}$. Here's the famous g=2 factor. So we are left with $(-2im \frac{\partial}{\partial_t} - \nabla^2 - eB_o.(\vec{L} + 2\vec{S}))\Psi = 0$ or – restoring the $\hbar$'s

$$(i\hbar \frac{\partial}{\partial_t} + \hbar^2 \frac{1}{2m} \nabla^2 + \frac{e\hbar}{2m} B_o.(\vec{L} + 2\vec{S}))\Psi = 0 \tag{1.6}$$

[1]Errors from ignoring the Vector potential squared term

It is worth checking what approximation is involved in ignoring the term in $O(eA)^2$. The move to the use of the covariant derivative $\partial_u \to D_u = \partial_u - ieA_u$ ignores the presence of an $\hbar$ term as $\bar{p} = \hbar\partial_u$ and eA_u both have units of momentum (being energy/velocity). Thus we should be substituting $\bar{p} = \hbar(\partial_u - \frac{eA_u}{\hbar})$. This is not obvious because we have put $c = \hbar = 1$

We want to form Hamiltonian energy terms viz $\frac{\bar{p}^2}{2m}$ (via $\frac{1}{2}mv^2$, classically). Multiplying out, we have a term $\hbar^2\partial_u \frac{eA_u}{\hbar}$ which becomes the spin Hamiltonian we want, viz $\frac{e\hbar B_o}{2m}$. Thus our ignored term $\frac{e^2A^2}{2m}$ must be small compared with this, or

$$\frac{e\hbar B_o}{2m} >> \frac{e^2 A_u^{\ 2}}{2m}$$

Or $\hbar B_o >> eA_u^2$. Now (and this is where the Physics comes in), we put $A_u \sim xB_o$ where x is some distance in the system. So we need to show that

$$1 >> \frac{ex^2 B_o}{\hbar}$$

Putting $B_0 = 10T, \hbar = 10^{-34}, e = 10^{-19}$, then $\frac{10^{-19}10^1}{10^{-34}}x^2 << 1$ so $x^2 << 10^{-16}, x < 10^{-8}M$

The question is what is x. It could be the size of the structure which contains the electron – i.e. the container (not OK). It could be the classical electron radius , $\frac{1}{2\pi\varepsilon_o}\frac{e^2}{m_e c^2}$, which is $3.10^{-15}M$ (OK) or it could be the scale distance in which the wave function decays.

g-2

Now we move to look at the correction to g – the “g -2” factor that heralded in the world of Quantum Field Theory – specifically Quantum Electrodynamics

This is taken from A.Zee’s “Quantum Field Theory in a nutshell” section III 6.

In what follows the “Gordon decomposition” is used to break up $\bar{u}(p')\gamma^{\mu}u(p)$ into a standard current part and a magnetic moment part.

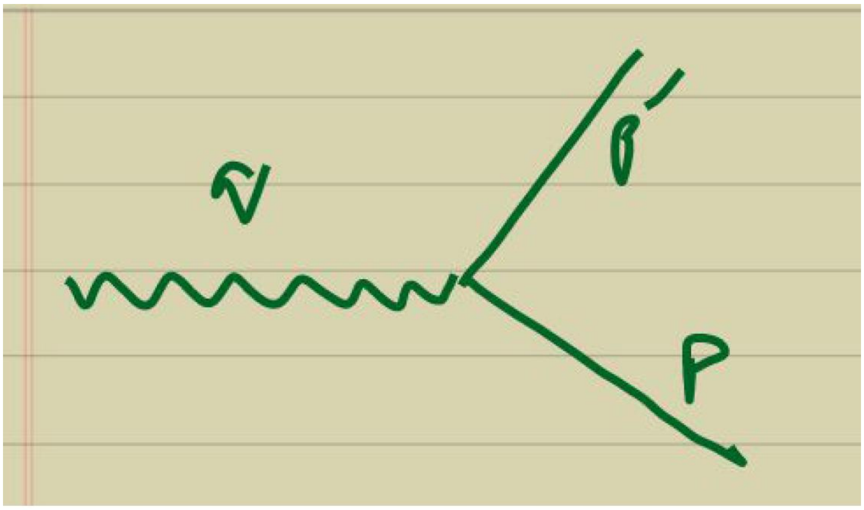
The tree level term – discovered by Dirac – has a γ^{μ} term. When more γ^{μ} terms are found in loop diagrams (and we are only looking at one loop diagram here – this is appropriate given that the Fine Structure Constant ,which determines the correction, is small and is raised to powers in more complex loops) then they all add (TBC) to both the current and magnetic terms. Thus they have the effect of changing the value of the charge – the coupling constant between the Dirac Electron Spinor field and the Electromagnetic field A_{μ} . Thus the g=2 value is not changed – that value is embedded into $\bar{u}(p')\gamma^{\mu}u(p)$ and remains as 2.

Of course experimentalists measure the actual value of ‘e’ which contains all of the loop terms and one can’t access the naked value of “e” – ie the value in the electron vertex.

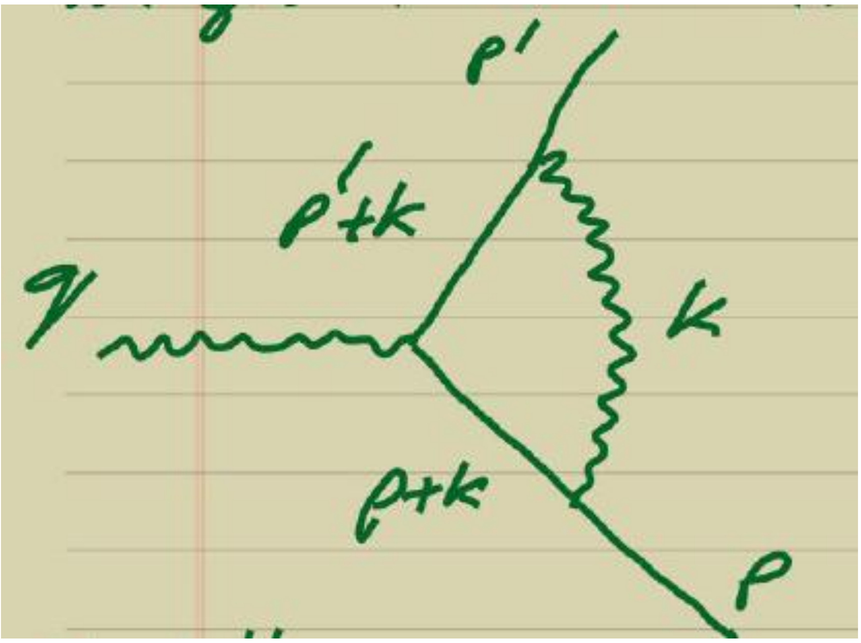
As $\gamma^{\mu} = (p+p')^{\mu}$ + magnetic moment terms, when we find $(p+p')^{\mu}$ terms in a loop diagram they equate to “anomalous magnetic moment” terms plus the (unobservable) γ^{μ} i.e. electron charge term. We search for these $(p+p')^{\mu}$ terms to get the anomalous magnetic moment terms.

The loop term

Zee start by looking at two Feynman diagrams together



This is the basic interaction terms with vertex $+ie\gamma^{\mu}$ and there are no internal propagators – so all lines (electron and photon) are real – they are 'on-mass-shell'. The additional Feynman diagram looks like.



(Zee states that other loop diagrams at this order only contain γ^{μ} terms). The second of these Feynman diagrams is more complex. There are three spin ½ vertices of the form

$+ie\gamma^{\mu}$

a photon propagator $i\frac{-g^{\mu\nu}}{k^2}$

and two spin ½ propagators $i\frac{\not{p}+m}{p^2-m^2}$

in and out of the main vertex.

Multiplying up and integrating over the appropriate four-momentum k, we have

$$\Gamma^{\mu} = \int \frac{dk^4}{(2\pi)^4}\frac{-i}{k^2}\left[ie\gamma^{\nu}\frac{i}{\not{p}'+\not{k}-m}\gamma^{\mu}\frac{i}{\not{p}+\not{k}-m}i\gamma_{\nu}\right]$$

What part of the Integral?

Now we need to know what part of this integral are we interested in:

Consider the Dirac electron current $\bar{u}(p')\gamma^{\mu}u(p)$ which equals (via the Gordon decomposition)

$$\bar{u}(p')\left[\frac{(p'+p)^{u}}{2m}+i\frac{\sigma^{\mu\nu}(p'-p)}{2m}\right]\bar{u}(p)$$

The form of the overall current[3] needs to be [See Zee III.6. equation 7] and putting $q=p'-p$

$$\langle p,s'|J^{\mu}(0)|p,s\rangle=\bar{u}(p')\left[\gamma^{\mu}F_1(q^2)+i\frac{\sigma^{\mu\nu}}{2m}q^{\nu}F_2(q^2)\right]\bar{u}(p)$$

[Invoking Lorentz invariance and current conservation]

And applying Gordon's decomposition to this we have – for the leading term in momentum transfer (i.e. $q^2=0$)

$$\bar{u}(p',s')\left[\frac{(p'+p)}{2m}F_1(0)+i\frac{\sigma^{\mu\nu}}{2m}q_{\nu}(F_1+F_2(0)\right]\bar{u}(p,s)$$

$F_1(0)$ is the electric charge – by definition 1: We can't get access to the naked charge. So we note that Dirac's magnetic moment has shifted by $\left[1+F_2(0)\right]$

Reforming the Integral

Returning to the Integral. We are now only interested in terms of the form $\dfrac{(p'+p)^{u}}{2m}$.

Multiplying through by $(\not{p}'+\not{k})-m$ and $(\not{p}+\not{k})-m$ we have for the [] term

$$\left[\gamma^{\nu}\frac{(\not{p}'+\not{k})-m}{(\not{p}'+\not{k})^2-m^2}\gamma^{\mu}\frac{(\not{p}+\not{k})-m}{(\not{p}+\not{k})^2-m^2}\gamma_{\nu}\right]$$

But – see A&H7p7 - $\not{p}^2$ (and therefore $\not{p}^2+\not{k}^2$) are actually p^2. So we come to

$$\Gamma^{\mu}=e^2\int\frac{dk^4}{(2\pi)^4}\frac{-i}{k^2}\left[\gamma^{\nu}\frac{(\not{p}'+\not{k})-m}{(p'+k)^2-m^2}\gamma^{\mu}\frac{(\not{p}+\not{k})-m}{(p+k)^2-m^2}\gamma_{\nu}\right]$$

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[3] My assumption here is that this is not a Dirac electron current, but an actual "real-world current" – with only these allowable parameters.

Zee reduces this to

$$\Gamma^{\mu} = e^2 \int \frac{dk^4}{(2\pi)^4} \frac{-i}{k^2} \frac{N^{\mu}}{D} \quad \text{where}$$

$$\frac{1}{D} = \left[\frac{1}{(p'+k)^2 - m^2} \frac{1}{(p+k)^2 - m^2} \right]$$

The Denominator calculation

Feynman has discovered that

$$\frac{1}{xyz} = \int_0^1 \int_0^1 \int_0^1 \frac{d\alpha . d\beta . d\delta}{(\alpha x + \beta y + \delta z)^3} \delta(\alpha + \beta + \delta - 1)$$

(which is why he was a genius) The second delta symbol here is a Dirac function, and enforces

$\delta = \alpha + \beta - 1$ when the third integral is performed giving

$$= \int_0^1 \int_0^1 \frac{d\alpha . d\beta}{(z + \alpha(x-z) + \beta(y-z))^3}$$ over the triangle bounded by $0 \le \alpha \le 1$ and $0 \le \beta \le 1$

In this case, putting $k^2 \to z$, $(p'+k)^2 - m^2 \to x$ and $(p+k)^2 - m^2 \to y$

We have

$$k^2 + \alpha((p'+k)^2 - m^2 - k^2) + \beta((p+k)^2 - m^2 - k^2)$$

And expanding out

$$k^2 + \alpha(p'^2 + k^2 - m^2 - k^2 + 2p'k) + \beta(p^2 + k^2 - m^2 - k^2 + 2pk)$$

Of course $p^2 = p^{\mu} p_{\mu}$ which is $E^2 - \overline{p}^2$ and the rest mass. As p is 'on-mass-shell', we then have $p^2 = m^2$, so we are left with

$$k^2 + 2\alpha p'k + 2\beta pk$$

Changing variable by putting $k \to L - (\alpha p' + \beta p)$ we get

$$L^2 + (\alpha p' + \beta p)^2 - 2L(\alpha p' + \beta p)^2 + 2L(\alpha p' + \beta p)^2 - 2(\alpha p' + \beta p)^2$$

and the middle two terms cancel to $L^2 - (\alpha p' + \beta p)^2$

Expanding this out and noting that the photon q, the photon four momentum is the difference between the incoming and outgoing electron momenta (i.e. $q = p' - p$), we have

$L^2 - (\alpha(p+q) + \beta p)^2$

Now some more multiplying out

$L^2 - \alpha^2(p+q)^2 + \beta^2 p^2 + 2\alpha\beta(p+q)p$

And further to

$L^2 - (\alpha^2(p+q)^2 + \beta^2 p^2 + 2\alpha\beta p^2 + 2\alpha\beta pq)$

But again $p^2 = m^2$

$L^2 - (\alpha^2(p+q)^2 + \beta^2 p^2 + 2\alpha\beta p^2 + 2\alpha\beta pq)$

$L^2 - (\alpha+\beta)^2 m^2 + \left[\alpha q + \alpha pq + 2\alpha\beta pq\right]$

What is in the square bracket is of order (q^2), and we will in future ignore

The Numerator calculation

Moving to the numerator N^μ , we move to replace k with L, so

$\gamma^\nu\left[\not{L} + \not{P}' + m\right]\gamma^\mu\left[\not{L} + \not{P} + m\right]\gamma_\nu$

where $\not{P}'^\mu \equiv (1-\alpha)p'^\mu - \beta p^\mu$ and $\not{P}^\mu \equiv (1-\beta)p^\mu - \alpha p'^\mu$

Now we multiply out in powers of the mass, m

$\gamma^\nu m\gamma^\mu\left[\not{L} + \not{P}\right]\gamma_\nu$ +

$\gamma^\nu\left[\not{L} + \not{P}'\right]\gamma^\mu m\gamma_\nu$ +

$\gamma^\nu m\gamma^\mu m\gamma_\nu$ +

$\gamma^\nu\left[\not{L} + \not{P}'\right]\gamma^\mu\left[\not{L} + \not{P}\right]\gamma_\nu$

Now we look at these in turn:

$\gamma^\nu m\gamma^\mu \not{L}\gamma_\nu$ & $\gamma^\nu \not{L}\gamma^\mu m\gamma_\nu$ terms are zero by symmetry over the integral

As $\gamma^\nu\gamma_\nu$ is 4, $\gamma^\nu m\gamma^\mu m\gamma_\nu$ is just a γ^μ term – which we ignore. Finally the

$\gamma^\nu \not{L}\gamma^\mu \not{P}\gamma_\nu$ and the $\gamma^\nu \not{P}'\gamma^\mu \not{P}\gamma_\nu$ terms are also single $\not{L}$ terms and integrate to zero: negative side of the integral cancels positive.

So we are left with

$$\gamma^{\nu} m \gamma^{\mu} \not{P} \gamma_{\nu} + \gamma^{\nu} \not{P}' \gamma^{\mu} m \gamma_{\nu}$$

which reduces to

$$m(\gamma^{\nu} \not{P}' \gamma_{\mu} \gamma_{\nu} + \gamma^{\nu} \gamma^{\mu} \not{P} \gamma_{\nu})$$

plus $\gamma^{\nu}\left[\not{L} + \not{P}'\right]\gamma^{\mu}\left[\not{L} + \not{P}\right]\gamma_{\nu}$

Here again we remove terms linear in $\not{L}$ which integrate to zero. So we are left with

$\gamma^{\nu} \not{P}' \gamma^{\mu} \not{P}' \gamma_{\nu}$ and $\gamma^{\nu} \not{L} \gamma^{\mu} \not{L} \gamma_{\nu}$, giving a total of 4 terms:

$m(\gamma^{\nu} \not{P}' \gamma_{\mu} \gamma_{\nu} + \gamma^{\nu} \gamma^{\mu} \not{P} \gamma_{\nu})$ plus $\gamma^{\nu} \not{P}' \gamma^{\mu} \not{P}' \gamma_{\nu}$ and $\gamma^{\nu} \not{L} \gamma^{\mu} \not{L} \gamma_{\nu}$

The mass terms “m”, can be reformulated thus – recalling that $\gamma^{\nu}\gamma_{\nu} = 4\mathrm{I}$
and also that $\gamma^{\nu} \not{a}\not{b} \gamma_{\mu} = 4ab$. So they become

$$= 4m((1-\alpha)p'^{\mu} - \beta p^{\mu} + (1-\beta)p^{\mu} - \alpha p'^{\mu})$$

$$= 4m((1-2\alpha)p'^{\mu} + (1-2\beta)p^{\mu})$$

As this equation is symmetric with respect to $\alpha \leftrightarrow \beta$ this becomes

$$= 4m(1-\alpha-\beta)(p'^{\mu} + p^{\mu})$$

We now turn to the quadratic terms. Firstly $\gamma^{\nu} \not{L} \gamma^{\mu} \not{L} \gamma_{\nu}$. Note that $\gamma^{\nu} \not{a}\not{b}\not{c} \gamma_{\nu} = -2\not{c}\not{b}\not{a}$.

So we have $\not{L} \gamma^{\mu} \not{L} = \gamma_{\sigma} L^{\sigma} \gamma^{\mu} \gamma_{\tau} L^{\tau}$

These two L terms, $L^{\sigma} L^{\tau}$, can be written as

$\dfrac{1}{4} g^{\sigma\tau} L^2$ and under the Integral $\int \dfrac{d^4 L}{(2\pi)^4} L^2$ is just a number. So we are left with a γ^{μ} term which is removed.

$\gamma^{\nu} \not{P}' \gamma^{\mu} \not{P} \gamma_{\nu}$ which is just $-2\not{P}\gamma^{\mu}\not{P}'$. (Note change of order as $\gamma^{\nu} \not{a}\not{b}\not{c} \gamma_{\nu} = -2\not{c}\not{b}\not{a}$)

Which further reduces $-2[(1-\beta)\not{p} - \alpha \not{p}']\gamma^{\mu}[(1-\alpha)\not{p}' - \beta \not{p}]$

Looking at the $\bar{u}(p')\alpha \not{p} u(p)$ term we get αm as – from Dirac –

$(\not{p} - m)u(p) = 0$ and $\bar{u}(p')(\not{p}' - m) = 0$ so we reduced to

$$-2[(1-\beta)\not{p}-\alpha m]\gamma^{\mu}[(1-\alpha)\not{p}'-\beta m]$$

Again the m^2 term gives a $+\gamma^{\mu}$ term which we can ignore and we have for the 'm' term

$$-2[(1-\beta)\not{p}]\gamma^{\mu}[-2\beta m]-2[-\alpha m]\gamma^{\mu}[(1-\alpha)\not{p}']$$

Now we note that $\gamma^{\mu}\gamma^{\nu}p'_{\nu}=2g^{\mu\nu}p'_{\nu}-\gamma^{\nu}\gamma^{\mu}p'_{\nu}$ (noting that the RHS term p'_{ν} has no Feynman slash). Expanding this, we have $=2p'^{\mu}+\not{p}'\gamma^{\mu}$

But, again $\bar{u}(p')\not{p}'=\bar{u}(p')m$ via the Dirac equation again. So we have a γ^{μ} term which we can ignore.

The $\not{p}\gamma^{\mu}$ term converts to $\gamma^{\nu}p_{\nu}\gamma^{\mu}$

$$=2g^{\mu\nu}p_{\nu}-\gamma^{\mu}p_{\nu}\gamma^{\nu}$$

$$=2p^{\mu}-\gamma^{\mu}\not{p}\ .$$

Pulling out $(\not{p}-m)u(p)=0$ yet again, we have yet another γ^{μ} term, which is thrown away. Collecting up terms, we have

$$+2m[(1-\beta)2p+\alpha(1-\alpha)2p']$$

Finally for the m^0 term we have

$$-2(1-\beta)\not{p}\gamma^{\mu}(1-\alpha)\not{p}'$$

$$-2[(1-\beta)(1-\alpha)\frac{1}{2}][\not{p}\gamma^{\mu}\not{p}'+\not{p}\gamma^{\mu}\not{p}']$$ (here we have divided and multiplied by 2)

We need Dirac again to generate an 'm' term. Recalling $(\not{p}-m)u(p)=0$, $\bar{u}(p')(\not{p}'-m)=0$ and

$$\gamma^{\nu}\gamma_{\nu}p'^{\nu}=4p'^{\mu}$$

$$\gamma_{\nu}p^{\nu}\gamma^{u}=4p^{\mu}$$

$$-2[(1-\beta)(1-\alpha)\frac{1}{2}][\not{p}4p'^{u}+4p^{\mu}\not{p}']$$

And now Dirac gives us the "m" for the $\not{p}$

$$-2[(1-\beta)(1-\alpha)]2m(p'+p)^{\mu}$$

Collecting all of these numerator terms terms:

$4m[1-\alpha-\beta](p'+p)^{\mu}$

$2m[\alpha(1-\alpha)+\beta(1-\beta)](p'+p)^{\mu}$

$2m[-2(1-\alpha)(1-\beta)]2m(p'+p)^{\mu}$

Taking a factor of $2m(p'+p)^{\mu}$ we have

$[2-2\alpha-2\beta+\alpha-\alpha^{2}+\beta-\beta^{2}-2+2\alpha+2\beta]$ or

$[\alpha-\alpha^{2}+\beta-\beta^{2}-2\alpha\beta]$ which can be factored as

$(\alpha+\beta)(1-\alpha-\beta)$

The main Integral

Now after all of this algebra, we return to the main integral

$$=-ie^{2}\int\frac{dL^{4}}{(2\pi)^{4}}\int\frac{d\alpha d\beta}{[L^{2}-(\alpha+\beta)^{2}m^{2}]}2m(p'+p)^{\mu}(\alpha+\beta)(1-\alpha-\beta)$$

Now the dL^{4} integral gives (rescaling by $(\alpha+\beta)^{2}$)

$$\int\frac{dL^{4}}{(2\pi)^{4}}\frac{1}{[L^{2}-m^{2}+i\varsigma]}=\frac{-i}{32\pi^{2}m^{2}}$$

Putting this back in we get

$$=-ie^{2}\int\frac{d\alpha d\beta.-i2m(p'+p)^{\mu}(\alpha+\beta)(1-\alpha-\beta)}{32\pi^{2}(\alpha+\beta)^{2}m^{2}}$$

$$=-e^{2}\frac{(p'+p)^{\mu}}{2m}\int\frac{d\alpha d\beta(\alpha+\beta)(1-\alpha-\beta)}{32\pi^{2}(\alpha+\beta)^{2}}$$

Recalling that this Integral is over the triangle bounded by $0\leq\alpha\leq1$ and $0\leq\beta\leq1$

It come to $\frac{1}{2}$ and we are left with

$$F_{2}(0)\frac{(p'+p)^{\mu}}{2m}=\frac{-e^{2}}{8\pi^{2}}\frac{(p'+p)^{\mu}}{2m}=\frac{\alpha}{2\pi}\frac{(p'+p)^{\mu}}{2m}\ .$$

So $F_2(0) = \frac{\alpha}{2\pi}$, with α being the Fine Structure constant, is the correcting factor to g. So g is not two but $2(1+\frac{\alpha}{2\pi})$ …. As measurements indicated.